Quantum Mechanics: Commutation

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I. Commutators: Measuring Several Properties Simultaneously

In classical mechanics, once we determine the dynamical state of a system, we can **simultaneously** obtain many different system properties (i.e., velocity, position, momentum, acceleration, angular/linear momentum, kinetic and potential energies, etc.). The uncertainty is governed by the resolution and precision of the instruments at our disposal.

In quantum mechanics, the situation is different. Consider the following:

1. We would like to measure **several properties** of a particle represented by a wavefunction.

2. Properties of a q.m. system can be measured experimentally. Theoretically, the measurement process corresponds to an **operator acting on the wavefunction.** The outcomes of the measurement are the eigenvalues that correspond to the operator. The operator is taken to be acting on a wavefunction that is either a pure eigenfunction of the operator of interest, or an expansion in the basis of functions.

In order to measure, for instance, 2 properties *simultaneously*, the **wavefunction of the particle must be an eigenstate of the two operators that corespond to the properties we would like to measure simultaneously.**

So, consider:

We have to operators, $\hat{A}$ and $\hat{B}$. Each operator acting on its eigenstate gives back A_i and B_j, respectively. If we have a wavefunction that is an eigenstate of **both** operators, then:

$$\hat{A}\psi_{A_i,B_j} = A_i\psi_{A_i,B_j}$$

$$\hat{B}\psi_{A_i,B_j} = B_j\psi_{A_i,B_j}$$

Thus,

$$\hat{B}\hat{A}\psi_{A_i,B_j} \;=\; \hat{B}A_i\psi_{A_i,B_j}$$

$$\hat{A}\hat{B}\psi_{A_i,B_j} \;=\; \hat{A}B_j\psi_{A_i,B_j}$$

So, using the fact that $\psi_{A_i,B_j} is an eigenfunction of \hat{A}$ and $\hat{B}$:

$$\hat{B}\hat{A}\psi_{A_i,B_j} \;=\; B_jA_i\psi_{A_i,B_j}$$

$$\hat{A}\hat{B}\psi_{A_i,B_j} \;=\; A_iB_j\psi_{A_i,B_j}$$

Subtracting the equations, we realize a compact notation for defining what is called a **commutator**:

$$[A, B] \;=\; \hat{A}\hat{B} - \hat{B}\hat{A}$$

For two physical properties to be simultaneously observable, their operator representations must commute.

Question: For a particle in a one-dimensional box, do the energy and momentum operators commute?

Answer: Since the particle-in-a-box wavefuntions are eigenfunctions of the total energy operator (the Hamiltonian) but not of the momentum operator, $-i\hbar\left(\frac{d}{dx}\right)$, the energy and momentum operators do not commute. Thus, these two properties cannot be known simultaneously given a particle-in-a-box wavefunction.

Heisenberg Uncertainty Principle

The Heisenberg Uncertainty principle is stated as:

$$\delta p \delta x \geq \frac{\hbar}{2}$$

For a quantum mechanical description of a particle's dynamics, we cannot know exactly and simultaneously both the particle's position and momentum. We must accept an uncertainty in measurements of these quantities as given by the inequality.

Quick justification:

Wave packet description of particle: single plane wave (momentum exact, position not known exactly) multiple plane waves (momentum inexact, position more precisely known)
To relate the uncertainty principle to variances and statistical measures, the relation:

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}$$

can be used in conjunction with wavefunctions and definitions of average properties:

$$\sigma_p^2 = < p^2 > - < p >^2 \quad (likewise\ for\ \sigma_x\ )$$