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Numerical Solutions for Transient and Nearly Periodic Waves in Shallow Water

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Abstract

This paper presents a study of several numerical methods for solving transient (time-dependent) model equations for waves in shallow water by means of the method of lines. The physical models studied include a time dependent mild-slope equation for narrow-banded linear waves in intermediate water depth, and nonlinear models for weakly dispersive long waves in Boussinesq and Green-Naghdi form. The models treat spatial dependence using second and fourth-order accurate finite-differences, and time integration is accomplished using a variety of methods including Euler predictor-corrector, fourth-order Runge-Kutta, and the Bulirsch-Stoer method using a modified midpoint scheme with polynomial extrapolation and adaptive step size.

Introduction

The problem of surface water wave propagation in slowly-varying domains continues to be of central concern in coastal engineering research. At present, the study of long, nearly non-dispersive waves has led to a number of successful time-stepping models which evolve an arbitrary wave field in space and time. For dispersive waves in intermediate water depth, no corresponding models which handle all frequencies simultaneously exist, except for the case of narrow frequency bands studied here. As a result, most of the previous work in intermediate depth waves has concentrated on the calculation of time-periodic wave fields and has thus neglected direct calculation of effects due to wave grouping and other types of motions which deviate from strict periodicity of the carrier wave.

In this study, we consider a variety of models governing unsteady wave propagation from the point of view of a two-equation system, with one equation being the usual depth-integrated mass continuity requirement and the second being the first integral of the horizontal momentum balance. This approach is new in the study of short surface waves in intermediate water depth. In nonlinear long wave theory, the approach has antecedents in the work of Wu and Wu (1982) and Miles and Salmon (1985), which are both important to the present discussion. We thus abandon the more standard three-equation, primitive-variable approach in favor of a two-equation form from which velocities would have to be obtained by differentiation.

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The Transient Mild Slope Equation

Smith and Sprinks (1975) first presented a time-dependent form of the mild-slope equation of Berkhoff (1972), given by

$$\tilde{\phi}_{tt} - \nabla_h \cdot (CC_g \nabla_h \tilde{\phi}) + (\omega^2 - k^2 CC_g)\tilde{\phi} = 0 \tag{1}$$

where $\tilde{\phi}(x, y, t)$, the velocity potential at the linearized free surface, is related to the velocity potential ϕ by

$$\phi = \frac{\cosh k(h+z)}{\cosh kh} \tilde{\phi} \tag{2}$$

and where $h(x, y)$ is the local, slowly-varying water depth. The remaining coefficients follow from linear wave theory; ω is the carrier frequency, C and C_g are the phase and group velocities, and k is the wavenumber. The parameters are related by the dispersion relation

$$\omega^2 = gk \tanh(kh) \tag{3}$$

where g is gravitational acceleration. If motion is assumed to be strictly periodic, with $\tilde{\phi} = \hat{\phi}\exp(-i\omega t)$, the model equation (1) reduces to the elliptic equation

$$\nabla_h \cdot (CC_g \nabla_h \hat{\phi}) + k^2 CC_g \hat{\phi} = 0 \tag{4}$$

given by Berkhoff (1972). Recently, several authors (Copeland, 1985; Madsen and Larsen, 1987; Panchang and Kopriva, 1989) have found that it is convenient to solve (2) in a time stepping format. Each paper makes use of an intermediate model which can be written as

$$\frac{C_g}{C}\frac{\partial \phi}{\partial t} + \nabla_h \cdot \mathbf{Q} = 0; \qquad \frac{\partial \mathbf{Q}}{\partial t} + CC_g \nabla_h \phi = 0 \tag{5}$$

where $\mathbf{Q}$ is a pseudo-flux vector. Eliminating $\mathbf{Q}$ from (3) and invoking strict periodicity recovers the model (2). The time-dependent model is in a convenient form for calculations, and a number of efficient numerical algorithms exist for its solution. However, (5) is not equivalent to (1), and does not properly maintain the distinction between phase and group velocities. It thus can not be used to study even weakly time-dependent wave fields despite the use of a time-stepping procedure.

We are interested here in solving problems involving modulated wave trains with corresponding narrow spectral bands. Model equation (1) is a suitable model for this case if the central frequency of the modulated wave train coincides with ω and the band width is not too large. Rather than solving (1) directly, we obtain an intermediate model in two-equation form. First, we write the local energy density associated with the velocity potential ϕ and surface displacement η as

$$\mathcal{H} = g\frac{\eta^2}{2} + \frac{CC_g}{g}\frac{(\nabla_h \tilde{\phi})^2}{2} + \frac{(\omega^2 - k^2 CC_g)}{g}\frac{\tilde{\phi}^2}{2} \tag{6}$$

We recognize that $\mathcal{H}$ is the Hamiltonian density with coordinate η and momenta $\tilde{\phi}$ as the conjugate variables. The evolution equations follow from

$$\eta_t = \frac{\partial \mathcal{H}}{\partial \tilde{\phi}} = -\frac{1}{g}\nabla_h \cdot (CC_g \nabla_h \tilde{\phi}) + \frac{1}{g}(\omega^2 - k^2 CC_g)\tilde{\phi} \tag{7}$$

$$\tilde{\phi}_t = -\frac{\partial \mathcal{H}}{\partial \eta} = -g\eta \tag{8}$$

Elimination of η between (7) and (8) recovers the second-order model (1). We retain the two first order equations instead in order to take advantage of the large array of methods available for solving first-order ODE's.

Numerical Methods

A large number of methods have been tested for solving the model equations (7) and (8). For the examples presented here, we have discretized $\tilde{\phi}(x,y)$ and $\eta(x,y)$ on a grid $\{x_i, y_j\}$. Spatial derivatives have been approximated with both second and fourth-order accurate finite differencing: the relative accuracy and stability of each method is undergoing further testing. After spatial discretization, we are left with a system of equations

$$\eta_{ij,t} = f_{ij}^{(1)}(t)$$
$$\tilde{\phi}_{ij,t} = f_{ij}^{(2)}(t) \tag{9}$$

which can be viewed as being a set of coupled ODE's (i.e., the "method of lines" approach). The exact form of $f^{(1)}$ depends on the discretization scheme used. The first-order ODE's are then integrated by any of a number of available methods; we have tested the Euler predictor corrector given by Wu and Wu (1982), the standard fourth-order Runge-Kutta algorithm with fixed step size, and the Bulirsh-Stoer extrapolation method together with adaptive step-size control, as presented in Press et al (1989). The numerical methods will be discussed more fully in an expanded version of the present report.

An Example: Wavemaking

The problem of generating waves in a long flume starting from a rest state serves as a fundamental test of the accuracy of the model in representing transient motion. We consider a one-dimensional domain $x \geq 0$, with $\tilde{\phi}_x$ specified at $x = 0$, corresponding to a prescribed wavemaker motion. The domain is taken to be initially motionless. At time $t = 0$, the wavemaker is started, and waves propagate down the length of the channel. Figure 1 illustrates two cases: one where the water is relatively deep and the group velocity is about half the phase velocity, and the other shallow, where the group and phase velocities are approximately equal. These results were obtained using second-order spatial differencing and the Euler predictor-corrector scheme. In Figure 1a, the wavelength to depth ratio is about 1:40, and waves are nondispersive. This result is clearly indicated by the common speed of the wave crests and the leading edge of the wave train. The leading edge of the wave train changes shape slightly due to the finite accuracy of the differencing and resulting numerical dispersion. Figure 1b shows the effect of difference between group and phase velocity in deep water, where the amplitude of an arriving wave train builds up slowly and where waves dissappear continuously at the leading edge of the wave train.

Models for Long Waves

The basic modelling scheme developed for the set of equations above provides a framework for any model equations that can be written in the form (7-8), with time derivatives on the left and space derivatives on the right. We have used this fact to establish modelling schemes for several additional problems, including principally the two-equation Boussinesq and Green-Naghdi models (Green and

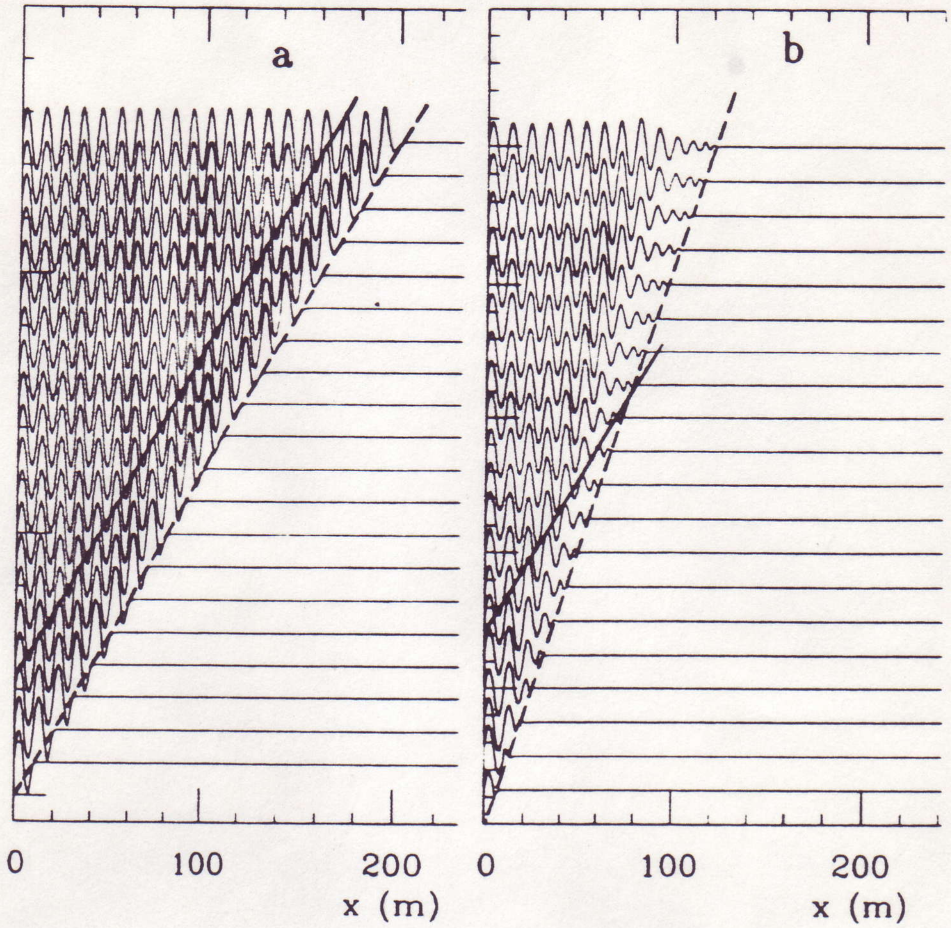

Figure 1: Transient linear wavemaker condition; shallow (a) and deep (b) water. Dashed lines indicate leading edge of wave train (group velocity); solid lines indicate crest trajectory (phase velocity).

Naghdi, 1976) developed by Miles and Salmon (1985). A two-equation system in the Boussinesq approximation has been studied previously by Wu and Wu (1982). Here, however, an alternate choice of equations leads to an explicit system after finite differencing, and we further include variable depth effects. The Green-Naghdi form of the governing equation is given by

$$\eta_t = -\nabla_h \cdot ((h+\eta)\nabla_h \tilde{\phi}) - \nabla_h^2 \left(\frac{1}{3}(h+\eta)^3 \nabla_h^2 \tilde{\phi} + \frac{1}{2}(h+\eta)^2 \nabla_h h \cdot \nabla_h \tilde{\phi}\right) + \nabla_h \cdot \left[\left(\frac{1}{2}(h+\eta)^2 \nabla_h^2 \tilde{\phi} + (h+\eta)\nabla_h h \cdot \nabla_h \tilde{\phi}\right) \nabla_h h\right] \quad (10)$$

and

$$\tilde{\phi}_t = -\frac{1}{2}(\nabla_h \tilde{\phi})^2 + \frac{1}{2}((h+\eta)\nabla_h^2 \tilde{\phi} + \nabla_h h \cdot \nabla_h \tilde{\phi})^2 - g\eta \quad (11)$$

A reduction of order made by retaining unknowns to second-order nonlinearity in nondispersive terms and linear order in dispersive terms yields the Boussinesq form. Various examples of the use of these models will be shown in the conference presentation.

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Appendix A: References

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