# Approximations of the Wave Action Equation in Strongly Sheared Mean Flows


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## Introduction

Rivers with strong freshwater runoff can lead to strongly stratified conditions that promote strong vertical current shear. (ex. Columbia River shown in figure 1) . Our goal is to develop a computational framework to support modeling of surface waves and resulting transport processes in strongly stratified and sheared environments. In particular, this study examines the accuracy of wave spectral models to account for the strong shear current effects on wave propagation.

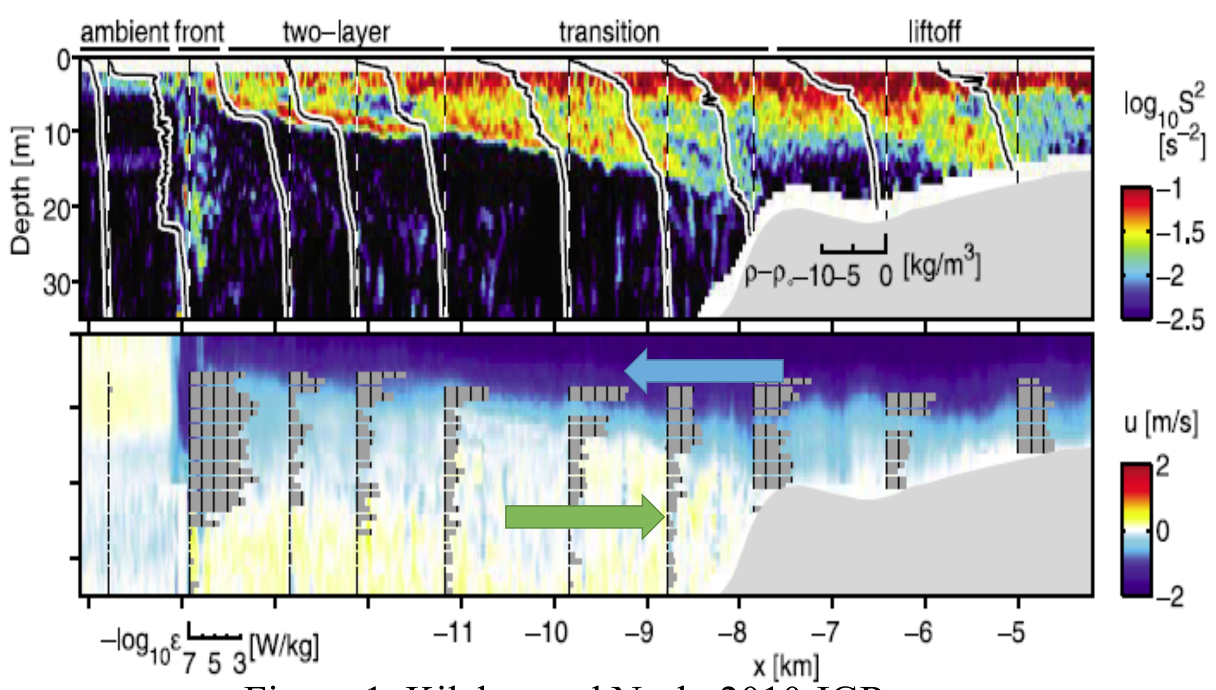


Figure 1. Kilcher and Nash, 2010 JGR

## Wave propagation modeling approach

Wave Spectral models solving wave action equation simplify the effects of currents on waves by assuming a *depth uniform current* in the dispersion relation as:

$$c=c_0+\bar{U} \qquad (1)$$

Where $c_0$ is the usual result for linear waves on a stationary domain as: $c_0=(\frac{g}{k}\tanh kh)^{1/2}$ (2)

and $\bar{U}$ is the depth average current.

Group velocity derived from the dispersion relation in this case would be :

$$C_{ga}=\partial\omega/\partial k=\partial(\sigma+k.\bar{U})/\partial k=\partial\sigma/\partial k+\bar{U} \qquad (3)$$

Skop, 1987 and Kirby and Chen, 1989 **(KC)** suggested a depth-weighted current $\tilde{U}$ resulting from the first order correction to the phase speed using perturbation solution to be used instead of the depth average velocity. $\tilde{U}$ Is defined as:

$$\tilde{U}=\frac{2k}{\sinh\ 2kh}\int_{-\infty}^{0}U(z)\cosh\ 2k(h+z)dz \qquad (4)$$

Where $U(z)$ is the depth dependent current, k is the wave number and h is the water depth. Gelfenbaum 2010 further simplified the expression by suggesting a depth-weighted current based on the peak frequency.

Following these definitions the absolute group velocity used in the models is clearly neglecting the dependence of $\tilde{U}$ on k. The corrected absolute group velocity would therefore be:

$$C_{ga}=\partial\omega/\partial k=\partial\sigma/\partial k+\tilde{U}+k\partial\tilde{U}/\partial k \qquad (5)$$

The term $k\partial\tilde{U}/\partial k$ is missing from these applications.

. To illustrate the importance of this neglected term two absolute group velocity comparisons are made in this study:

1) Linear sheared current
2) Using ebb tide current profile measured in Columbia River.

## Group Velocity comparison for linear shear current

Consider a linear shear current with the velocity distribution:

$$U(z)=U_s(1+\alpha\frac{z}{h}) \quad \text{Where } \alpha=\Omega\frac{h}{U_s} \qquad (6)$$

Which is shown in figure 2.

The absolute group velocity presented by Thompson (1949) and Biesel (1950) , is found to be:

$$C_{ga}=U_s+\frac{g(1+G)-\Omega C_{rs}G}{2g-\Omega C_{rs}}C_{rs} \qquad (7)$$

Where: $$G=\frac{2kh}{Sinh2kh} \qquad (8)$$

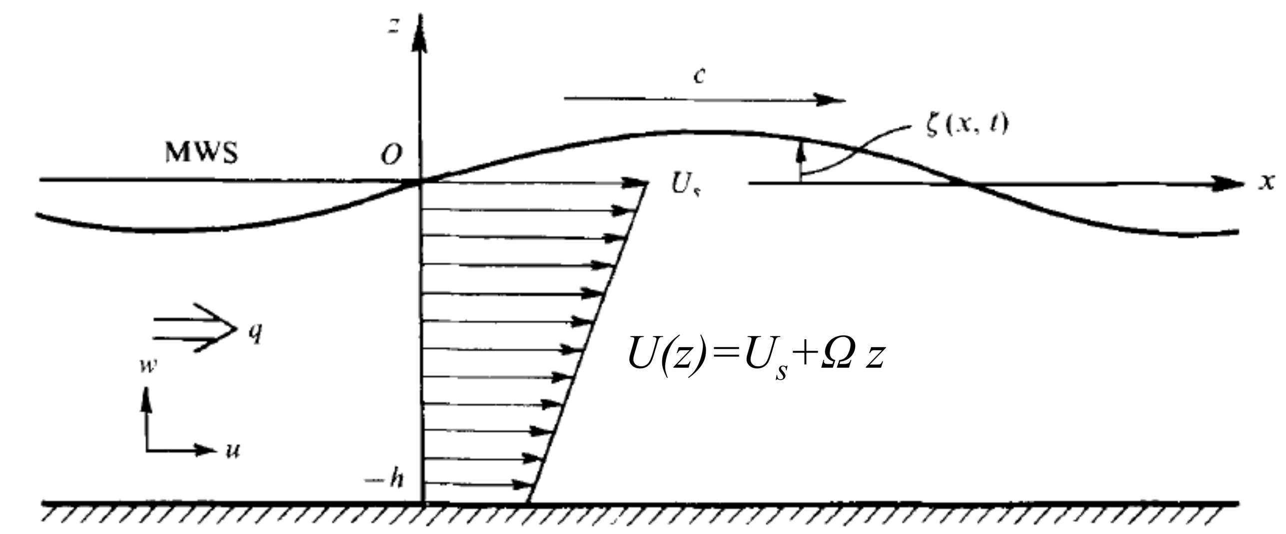


Figure 2. Definition sketch (Jonnson *et al* 1978 J. Fluid Mech)

First and second order perturbation solutions relatively result in the absolute group velocities as:

$$C_{ga}^{KC1}=C_{gr}+\tilde{U}+k\partial\tilde{U}/\partial k \quad (9) \qquad C_{ga}^{KC2}=C_{gr}+\tilde{U}+c_2+k\partial\tilde{U}/\partial k+k\partial c_2/\partial k \quad (10)$$

Absolute group velocity based on depth averaged and weighted current :

$$C_{ga}^{d\_avg}=C_{gr}+\bar{U} \quad (11) \qquad C_{ga}^{d\_wgt}=C_{gr}+\tilde{U}\ (\text{or } \tilde{U}_p) \quad (12)$$

$\tilde{U}$ : first order order correction to phase speed defined as the depth weighted current **(KC)**

$C_2$ :second correction to phase speed (**KC**).

$C_{gr}$: Relative wave group velocity as $C_{gr}=\partial kc_0/\partial k=\partial\sigma/\partial k=\frac{C_0}{2}(1+G)$ and $\tilde{U}_p$ is the depth weighted current based on peak frequency as suggested by Gelfenbaum.

Introducing α defined in (6) and the Froude number as $F=\frac{U_s}{\sqrt{gh}}$ plots of the absolute group velocity comparison are shown in figure 3 for various choices of α and F.

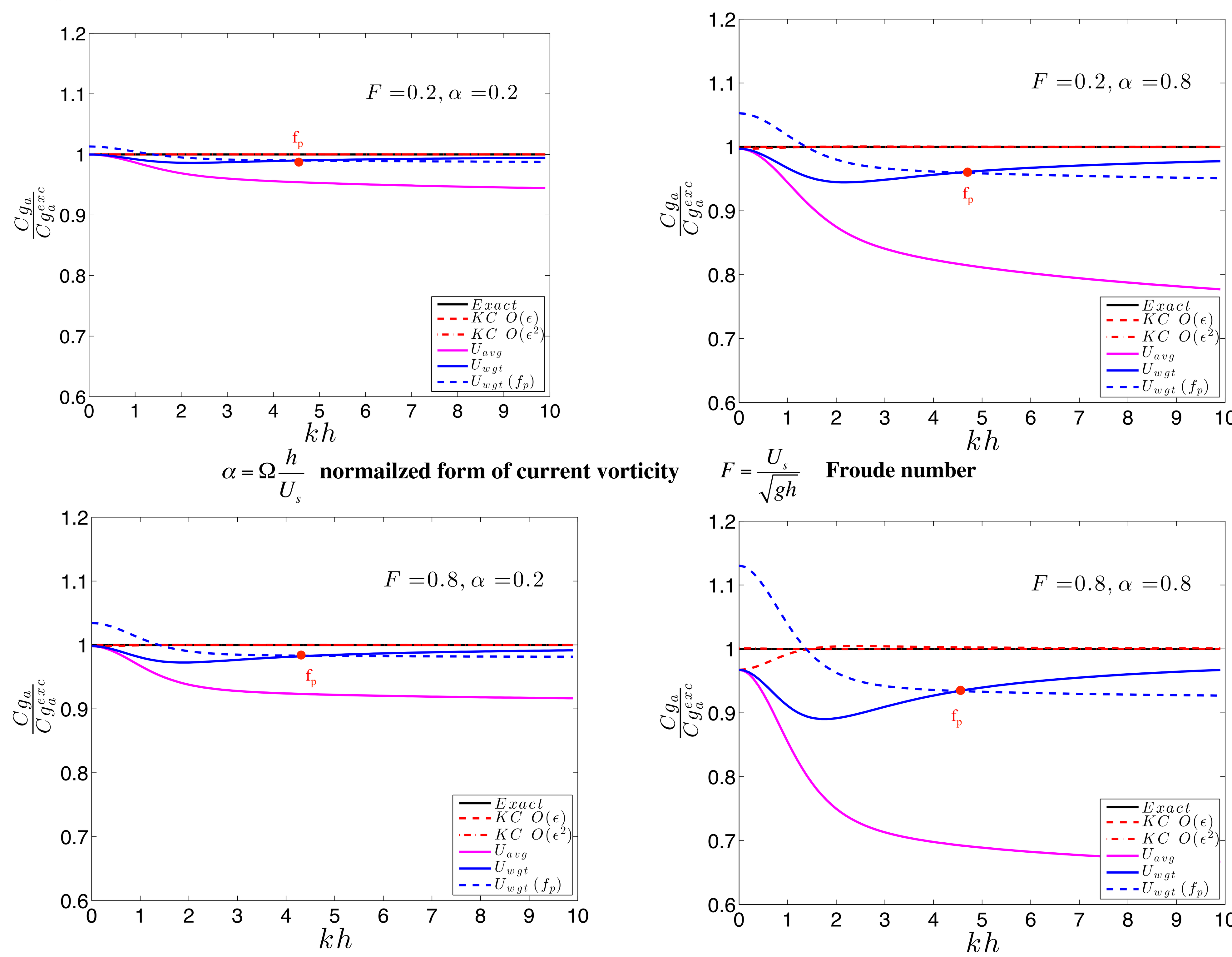


Figure 3. Absolute group velocity comparison: Linear shear current

## Group Velocity comparison for a given current velocity profile

To give a general idea of how strong currents are in real world, a current velocity measured at Columbia river mouth was used. Figure (4) shows the current profile and the 6[th] order polynomial fitted in order to do numerical calculations. The current profile used is from RISE project measured by Ocean Mixing Group, Oregon State University .

- Following Dong and Kirby (2012) a shooting method is adapted to solve the Rayleigh equation numerically.
- The first and second order perturbation method are also calculated numerically

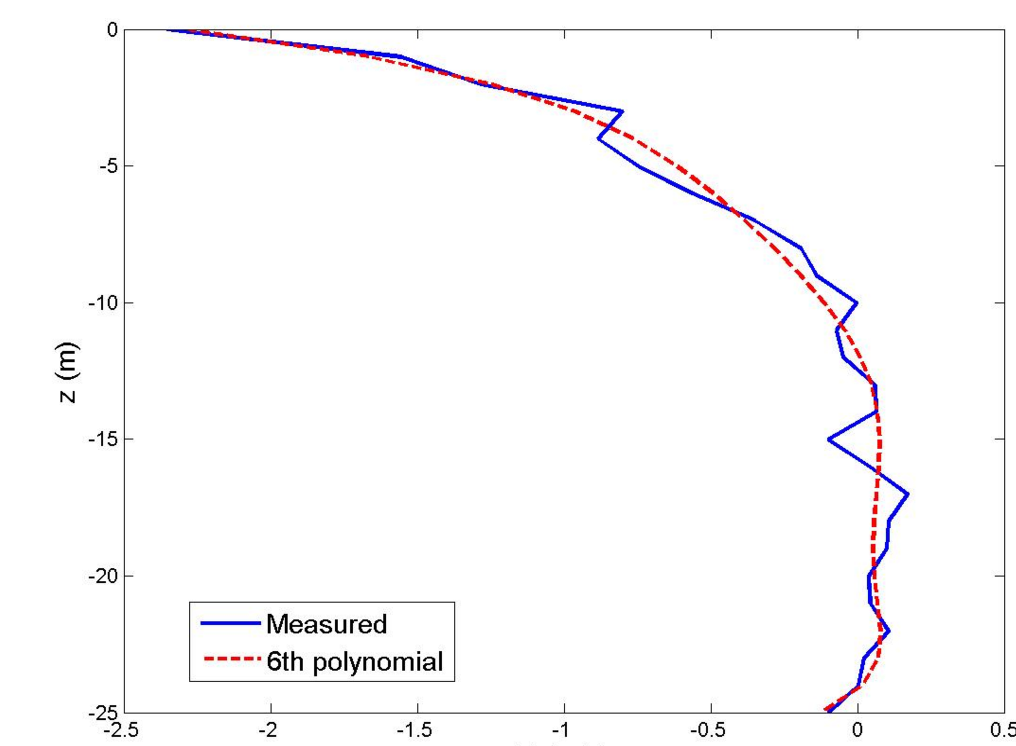


Figure 4. Columbia River current profile during ebb tide (Dong and Kirby ICCE 2012)

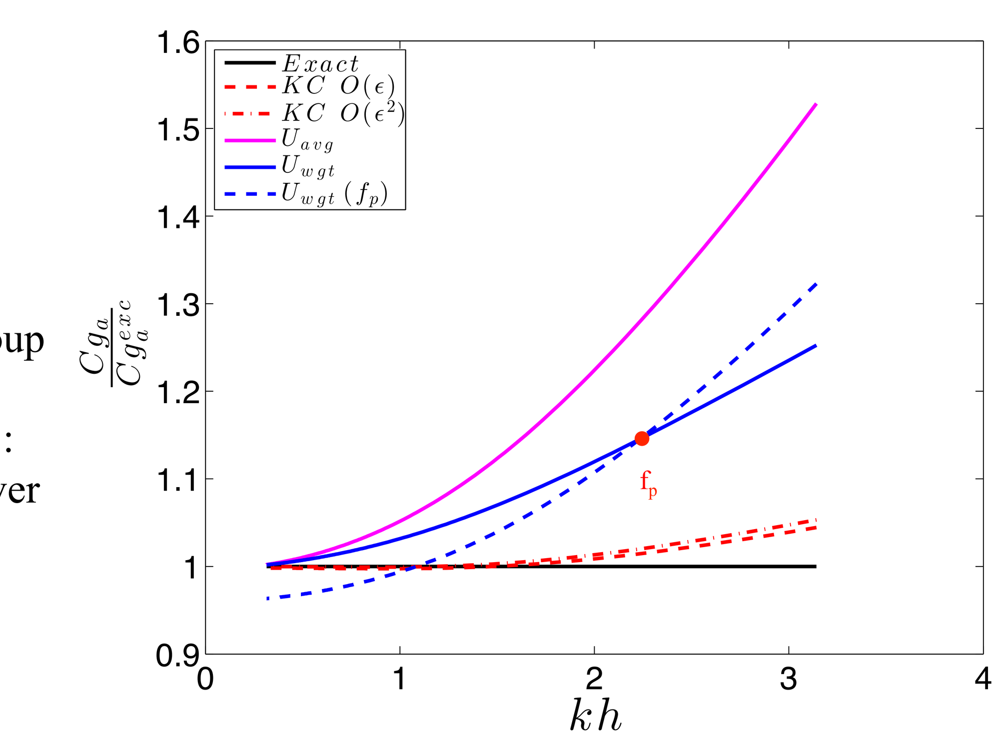


Figure 5. Absolute Group velocity comparison : Columbia River

## Conclusions

- Figures 3 and 5 show the resulting error by neglecting the term $k\partial\tilde{U}/\partial k$. Although the depth weighted current without the term is a better approximation to the depth averaged current, an error up to 20% still appears in the absolute group velocity from deep to shallow water.

## References


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