A Modest Proposal for Reconceptualizing the Activity of Learning Mathematical Procedures

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The Need for Reconsidering Conceptual and Procedural Knowledge

The values of conceptual and procedural knowledge in mathematics, and their respective roles in learning and applying and teaching mathematics, have been debated for a century or more (Hiebert, 1986; Resnick & Ford, 1981). Continued attention to these issues is explained, in part, by the persistent belief that these descriptions of knowledge capture something fundamental about knowing mathematics and, in part, by the increasing sophistication with which learning processes are being described and the consequent ability to sharpen the definitions of, and distinctions between, conceptual and procedural knowing.[1]

Despite the historical tendency to value one kind of knowing over the other, the current consensus view is that both are essential in becoming mathematically proficient (National Research Council, 2001). It is not a question of which is more important but how these kinds of knowing interact. How do they support, or interfere with each other as students study mathematics?

One way to make progress in describing the potential relationships between conceptual and procedural knowing is to examine, in detail, particular cases of teaching and learning. We believe that important relationships are found in the particulars as teachers present tasks and ask questions, and as students attempt to solve problems and respond to questions. In this paper, we consider one eighth-grade mathematics lesson. On the surface, it appears to be a lesson on learning procedures but a further analysis suggests a complex interplay between conceptual and procedural activity. We focus on this interplay because we believe it provides some insights into a coordination, rather than competition, between conceptual and procedural learning. We argue that understanding the details of this kind of classroom mathematical discussion is central for improving mathematics teaching.

An Intriguing Result from the TIMSS 1999 Video Study

As part of the 1999 Trends in International Mathematics and Science Study, nationally representative samples of eighth-grade mathematics lessons were filmed in seven countries (Australia, Czech Republic, Hong Kong SAR[2], Japan, Netherlands, Switzerland, and United States). Analyses of the lessons focused on features of teaching that might lead to differences in students' opportunities to learn mathematics (Hiebert et al., 2003). One feature examined carefully was the kind of mathematics problems presented to students (Smith, 2000; Stein, Grover, & Henningsen, 1996; Stein & Lane, 1996). Specifically, problems were coded based on the statements of the problems, whether made by the teacher or the textbook or another source. Problems, as presented, indicate the apparent intent of the problem. All presented problems were placed into one of three categories:

- Stating Concepts: Recalling mathematical properties or conventions (e.g., "What are the properties of an equilateral triangle?" Or "Plot the point (3, 2) on the coordinate plane.")

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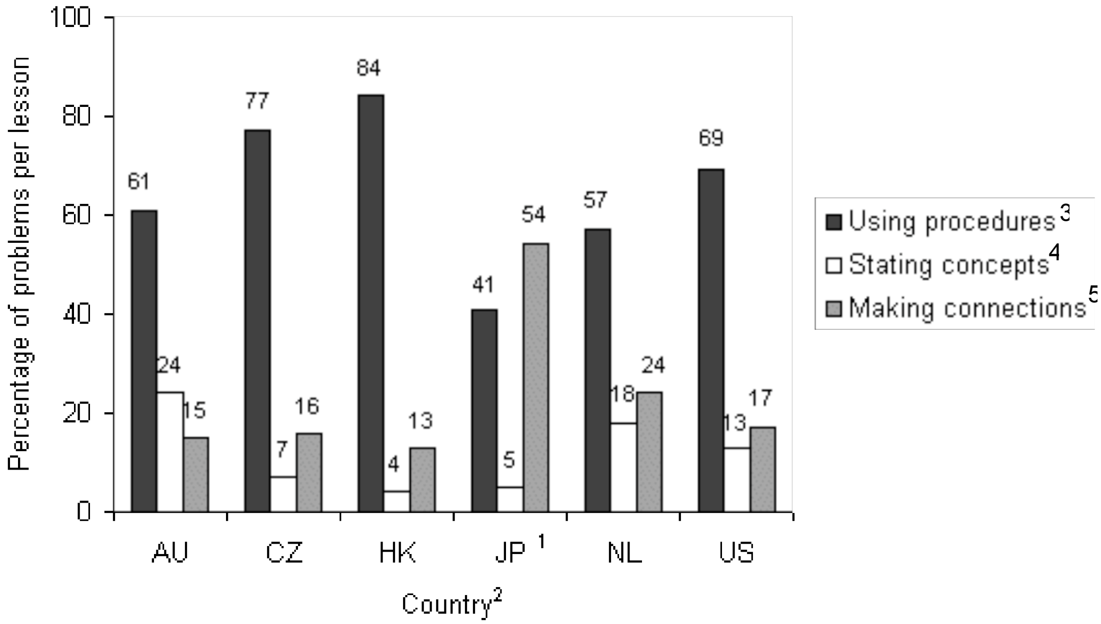
- Using Procedures: Solving problems using known procedures (e.g., "Solve for x in the equation 2x + 3 = 7".)
- Making Connections: Constructing relationships among mathematical ideas, facts, or procedures. Engaging in conjecturing, justifying, generalizing, and so on (e.g., "What is different about the equations 2x + 3 = 7 and 2(x + 3) = 2x + 6?")

Results showed that, on average, lessons in Japan and Hong Kong SAR were at opposite ends of the spectrum with respect to the kinds of problems presented (Hiebert et al., 2003). As shown in Figure 1, 84% of the problems per lesson in Hong Kong SAR were presented as Using Procedures whereas 41% of the problems per lesson in Japan were presented as Using Procedures. The majority of problems per lesson in Japan were presented as Making Connections, the kind of problem associated with conceptual activity.

Figure 1. Average percentage of problems per lesson presented as each type.



[1]Japanese mathematics data were collected in 1995.
[2]AU=Australia; CZ=Czech Republic; HK=Hong Kong SAR; JP=Japan; NL=Netherlands; and US=United States.
[3]Using procedures: CZ>JP, NL; HK >AU, JP, NL, US; US>JP.
[4]Stating concepts: AU>CZ, HK, JP; NL, US>HK, JP.
[5]Making connections: JP>AU, CZ, HK, US.
NOTE: All reported country differences are significant at p<.05. Analyses only include problems with a publicly presented solution. Analyses do not include answered-only problems (i.e., problems that were completed prior to the videotaped lesson and only their answers were shared). English transcriptions of Swiss lessons were not available for these analyses. Percentages may not sum to 100 because of rounding.

The difference between Hong Kong SAR and Japan in the types of problems presented is especially intriguing because these are the two highest-achieving countries in the sample. How is it that the teaching approaches in two high-achieving countries have made such different choices with regard to an issue that mathematics educators in the United States often consider critical for high quality teaching—the relative emphasis on conceptual versus procedural activity?

These results generate alternative hypotheses regarding the conceptual versus procedural issue.

1. Perhaps the hotly contested issue surrounding conceptual and procedural knowing is not so important. Student achievement might be driven by factors other than whether teachers emphasize concepts or procedures.
2. Perhaps the coding in this study did not capture what is critical in procedural and conceptual activity. Maybe the traditional distinction between conceptual and procedural is too crudely defined and starkly contrasted to capture what really is intended and/or what is operationalized as procedural problems play out in classroom activity.

In this paper, we would like to pursue the second of these hypotheses—that the constructs of procedural and conceptual knowing need to re-examined with an eye toward more precisely describing the possible interactions between them during classroom practice. Although the video data cannot confirm or deny the hypothesis, examples from the Video Study provide a unique opportunity to conduct further analyses into the nature of procedural activity.

As a vehicle for our reconceptualization of procedural learning, we will use an eighth-grade mathematics lesson from Hong Kong SAR. The goal is to understand the "procedural" character of this lesson. In the end, we will claim that the mathematical analyses engaged in by the teacher and students do not fit the traditional and simple definitions of procedural (or conceptual). Understanding this kind of knowing, we believe, sheds light on the conceptual/procedural issue and suggests the importance of defining more precisely the constructs used to describe mathematics learning.

The Case of Hong Kong SAR

Background

The finding of a procedural emphasis in Hong Kong SAR eighth-grade mathematics teaching is not an artifact of the coding definitions used in the TIMSS Video Study. Similar observations have been made by others, including Hong Kong SAR mathematics educators about the country's mathematics classroom practices (Leung, 1995, 2001). But what, exactly, is meant by "procedural" when this term is used by different authors to describe mathematics instruction?

U.S. mathematics educators share a sense of what is intended by this term. In the United States, the traditional connotations of procedural often include one or more of the following characteristics:

- Executing a sequence of known steps of a particular solution method to move from the problem statement to the answer.
- The steps often prescribe how to move and transform written symbols according to well-defined rules.
- Little or no reasoning is required to execute procedures successfully; practice and memory are key.

- Efficiency and accuracy are primary goals, and these are achieved by practicing the procedure on similar sets of problems; little or no modification of the procedure is required.

Does “procedural” carry the same connotations in Hong Kong SAR? Several clues suggest it does not. This is important because it would indicate that more complete and precise definitions are needed to communicate about these constructs. This, in turn, would open the door to reconsidering the traditional distinctions between procedural and conceptual.

A first clue that procedural in Hong Kong SAR classrooms means something different than it does in U.S. classrooms comes from the TIMSS achievement scores. Eighth-grade students in Hong Kong SAR performed very well on the test, a test that includes a number of conceptually-demanding items. Although not impossible, it is difficult to imagine that high performance on this test would be associated with an emphasis on the relatively narrow U.S. approach to procedural activity.

A second clue that something different than traditionally-defined procedural activity is happening in Hong Kong SAR classrooms is provided by subjective impressions gained from watching the videotapes of Hong Kong SAR lessons. There is no doubt that students are studying procedures. But there seem to be many subtle differences between what these students are experiencing and what is captured by the bulleted characteristics identified above. Indeed, it is these differences that form the focal point for this paper and that we will describe momentarily.

A third clue that something different than a U.S. version of “procedural” is occurring in Hong Kong SAR classrooms is found in the literature. Expanding temporarily the view to include other Asian countries, Kobayashi (1984, p. 110) notes that repetition and “rote” learning are important aspects of Asian curriculum and pedagogy, but “this rote is not really repetition since, from the practitioner’s point of view, each ‘repetition’ is regarded as always containing something new.” Similarly, Leung (2001, p. 40) observes that “repetitive learning” in Asian educational cultures “is continuous practice with increasing variation.” Moreover, this kind of repetition can “lead to deep understanding.” Although these observers label the activity as procedural, they apparently have something in mind other than the rather simplistic notion of procedural common in the United States. In order to examine these differences further, we describe an eighth-grade lesson in Hong Kong SAR.

A Classroom Lesson

Our analysis of the lesson is presented in several parts. First, we present a few excerpts from the transcript that convey a sense of the sequence of problems presented and questions asked during the first two-thirds of the lesson. Second, we present the full transcript for the culminating six-minute segment of the lesson. We then use the excerpts of the lesson to describe several features of the procedural character of the lesson.

Lesson flow. The eighth-grade lesson focuses on calculating squares and square roots. As the lesson proceeds, the discussion begins zeroing in on the effects of squaring and taking square roots on the sign-value of numbers. Some excerpts from the lesson provide a sense of its flow. The lesson was conducted in English so the transcript is a direct transcription.

```
00:00:06:23 T     Okay, page 161.  Uh, we have learned something about the
square, do you remember?
00:00:14:11 SS    No.
00:00:15:01 T     No?  So uh, we have learned- for example, I give you a...
question for you to think about.
00:00:22:14 T     So first of all, what is... the square... of three?

00:00:48:12 S     Nine.
00:00:49:14 T     Nine, yes.  So uh, we can answer the question like that-
the square of three... is equal to nine- nine.

00:01:40:07 T     So that it- the meaning of uh, square of three is equal to
three times three.  So the answer is nine, okay?
00:01:47:17 T     So, what is the square of negative three?

00:03:23:22 SN    Nine.
00:03:24:12 T     Nine.
00:03:27:05 T     Okay, now uh, we will learn another- another type of
question.
00:03:56:04 T     Okay.  Now, what is the positive value of A if the square
of A is nine?

00:04:48:01 T     So then now, we want to find A.  So some of you say, the
answer is three.
00:04:53:19 T     So I- I think you can go- oh three times three is nine, so
the answer is three.
00:04:59:00 T     Uh, I will teach you a step to solve.  So, uh, if you find
the- uh, the value of A, you can write like this.
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The teacher then writes on the chalkboard:

$a^2 = 9$

$a = \sqrt{9}$

$a = 3$

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00:05:21:29 T     So what is the meaning of this radical sign?  So the
meaning is the same as this sentence.  Okay?
00:05:29:18 T     Square root of nine means to find a positive value of A

00:06:02:14 T     What is the negative... value of A if the square of A is
nine?
00:06:17:01 T     So the result is the same, the square of A is nine.  But
now, you want to find the negative number.

00:07:03:28 T     What is the difference?  What is the difference between
this part and the- and the previous question?
00:07:18:08 T     But now A, A is a negative number, not negati- negative A.
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The teacher writes on the chalkboard:

$a^2 = 9$

$a = -\sqrt{9}$
$a = -3$

```
00:09:05:08 T     So if you want to find the negative value, then in- in
front of the, uh, square root-
00:09:12:28 T     Before the square root, you write a negative sign.
00:09:16:06 T     So that you can get the answer is negative three.  Okay?
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The teacher then assigns a number of short practice exercises similar to those above. After students volunteer the answers, the teacher returns to the above cases:

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00:14:06:04 T     So you must know that, if uh, a- A squared is equal to
nine, there is two- sorry, there are two answers.

00:14:13:12 T     There are two answers... to make uh, A squared equal to
nine.
00:14:17:12 T     The values of A may be positive three or negative three.
Why?
00:14:23:10 T     Because negative three times negative three is nine.  And
three times three is also equal to nine.  Okay?
```

A few additional exercises are assigned (e.g., find the two solutions for the square root of 64). After students respond with the answers, the teacher begins the final episode that includes the most challenging problems of the lesson.

<u>The culminating episode</u>. The transcript for the episode is printed below.

```
00:22:28:18 T     Okay, think... think about, uh, square root of negative
four- sorry, square root of the square of negative four.
00:22:37:22 T     Is it equal to four or equal to negative four?
00:22:48:10 T     Lo Chi Hung.
00:22:49:18 SN    Four.
00:22:50:10 T     Four.  Why?
00:22:52:00 SN    Four
00:22:53:16 SS    Four.  Four.
00:22:55:23 SN    Negative four times negative four is- it is equal to 16 and
square root of 16 is four.
00:23:00:24 T     Yes, very good.
00:23:04:12 T     Yes, very good.
00:23:05:10 T     So, this square root of the square of negative four, you
can express-
00:23:13:00 T     the number inside, you can expand it as- you can calculate.
00:23:17:05 T     Negative- the square of negative four is equal to negative
four times negative four.
00:23:22:09 T     So it is equal to positive 16.
00:23:26:18 T     So in this case, the answer...
00:23:30:06 T     Is it equal- equal to negative- negative four?
00:23:32:20 SN    No.
00:23:33:24 T     Why?  Because there is no negative sign in the front.  So,
the answer must be positive, okay?
00:23:42:26 T     So in this case, you can see that the answer is positive.
00:23:47:21 T     You can- you can find the answer is positive if there is no
sign.  Okay?
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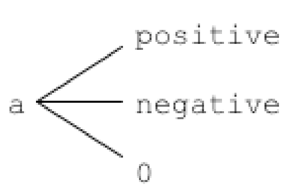
00:23:53:11 T If the answer is negative, only if there is a negative sign, uh, before the square root, okay?
00:24:04:10 T So in this case, there is no negative- negative sign...
00:24:08:06 T Uh, uh, in the front, so, we call- uh, we can find the answer positive. We can find a positive answer.
00:24:18:01 T Do you have any questions?
00:24:20:11 T Okay. Now, uh, we have another question for- for- uh, for you to think about. Uh, it is this.

The teacher writes on the chalkboard: $\sqrt{\square 4}$

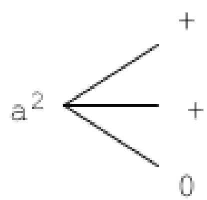
00:24:51:08 SN No solution.
00:24:51:12 T Okay.
00:24:52:11 SN No solution.
00:24:57:08 T Okay, wait a min- wait a minute for you to think about this.
00:25:02:27 T So, this, uh, square root of a- sorry, uh, negative- sorry, square root of negative four.
00:25:09:22 T Is it equal to two, or negative two, or no solution?
00:25:13:17 SS No solution, no solution.
00:25:16:18 T Okay. Who says that it is equal to two?
00:25:19:12 SS [Laughter]
00:25:19:28 T Put up your hand.
00:25:24:03 T Why? Because take square root means- what is the number times itself equal to negative four?
00:25:31:04 T So two times two is four.
00:25:33:13 T So it- it does not equal negative four. It is- it does not equal negative four.
00:25:38:27 T How about negative two? So this answer is incorrect. How about this?
00:25:45:02 T Is it correct?
00:25:46:06 SS [Laughter]
00:25:46:14 T Do you think this is correct? Please put up your hand.
00:25:49:29 T Why? Because negative two times negative two... equals...?
00:25:56:04 SS Four.
00:25:56:22 T Four. So it does not equal negative four.
00:26:00:07 T So, there is...?
00:26:02:13 SS No solution.
00:26:02:25 T No solution.
00:26:04:10 T Why? Because, uh, if you find that it is, uh, just like that, A squared is equal to negative four.
00:26:11:04 T So which number times itself is equal to a negative number?
00:26:15:02 SS None.
00:26:15:15 T None. Because a number A- all number can be, uh, divided or grouped into three- uh, must be one of these.

The teacher writes on the chalkboard:



00:26:23:23 T One is positive.
00:26:26:23 T One is...?
00:26:27:25 SS Negative.
00:26:28:13 T Negative. Or...
00:26:29:26 SN No solution.
00:26:31:07 T No solution. No.
00:26:33:07 T A number must be one of- one of- one of these. Maybe it is a positive. Maybe it is negative, or...?
00:26:42:15 SS Zero.
00:26:43:08 T Zero, yes. Very good.
00:26:44:24 T So uh, if A squared- we see, uh- by case, so if A is positive, then what is the value of A squared?

The teacher writes on the chalkboard:



00:26:54:21 T Positive? Negative? Or zero?
00:26:58:21 SS Positive.
00:26:59:15 T Positive. And then if it is negative, what is the result of the square of A?
00:27:05:00 SS Positive.
00:27:05:16 T Positive. If it is zero, then what is the squ- square of zero?
00:27:09:03 SS Zero.
00:27:10:07 T Zero. So is there any answer equal to negative?
00:27:13:17 SS No.
00:27:14:07 T No. So, this answer- this neg- uh, negative four, you cannot find the answers. Okay?
00:27:21:06 T Because, all the square...
00:27:24:26 T All the square, you cannot find the negative result.
00:27:28:22 T Do you have any questions? No.
00:27:31:10 T So uh, in this case, we just, uh, learned something about the meaning of the square root.
00:27:37:05 T So that's- what is this meaning of square root?
00:27:39:14 T So, take square root of nine is to find the number of A such that the square of A is equal to nine.
00:27:47:12 T So you just take the square root of nine to find the answer.
00:27:52:00 T And then, uh, if A squared is equal to A, then there are two values.
00:27:57:00 T So, in this case, it solves the answer is- the answer is positive.
00:28:01:18 T If this one, there is a negative sign, then the answer is negative.
00:28:06:26 T Okay?
00:28:08:03 T Finally, there are some, uh, uh, special, uh, questions for you to think about.
00:28:14:08 T So you must think carefully that, uh, this answer must be positive because there is no negative sign.
00:28:21:29 T This one... because inside there is a negative number, then it is- it is, uh- there is no solution.

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00:28:31:23 T     Okay, if there is no, uh, questions,
00:28:33:23 T     uh, please go home to read page 163 to page 164.
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The lesson ends a few minutes later.

The nature of the procedural activity. Looking at the flow of tasks and discourse across the lesson, the notion of repetition with variation (cited above) seems to be an appropriate description. But it is not just any kind of repetition and variation. The claim we will elaborate here is that the topic around which the repetition was executed and the kind of variations introduced were carefully selected and sequenced to develop both procedures and concepts simultaneously.

Our interpretation of the lesson suggests several key ingredients that enable productive interplay between procedural and conceptual activity. A first ingredient is *a mathematical topic that is of an appropriate size and richness*. Squaring and square roots is a topic that can be explored within a single lesson if the numbers are friendly and students can focus on the operations. The inverse nature of squaring and taking square roots affords a number of useful connections if they are considered together rather than treated separately in different lessons. In addition, there are a number of predictable misconceptions or mistakes that students make when encountering this topic for the first time. An example is the often overlooked difference between $\sqrt{9}$ and $\sqrt{-9}$. These anticipated errors can guide decisions about the sequence of tasks and issues to be probed during the lesson.

In fact, the 'culminating episode' focuses on one problem: $\sqrt{-4}$. This one problem is used as a backdrop for a systematic exploration of the meaning of square root. To put it another way, the procedure itself fades to the background while the reasoning process is brought to the foreground. Note how the teacher explicitly breaks down the reasoning process involved in thinking of square roots (and squares) into three cases: (1) a = positive → a^2 = +; (2) a = negative → a^2 = +; (3) a = 0 → $a^2 = 0$. From there, she makes the argument that for no value of "a" will "a^2" be negative.

It might be argued that the one procedural problem of $\sqrt{-4}$ is acting as an entryway into a deeper discourse on the relevant reasoning process involved in thinking of the interrelation of the square root to the squaring process for signed numbers. Some mathematical topics, it seems, more easily allow for such a manner of 'entering into' the underlying reasoning implicit in procedures. Other topics might not offer the richness of underlying concepts that are necessary for such probing.

A second ingredient that enables repetition with variation in a way that might lead to conceptual understanding is *a carefully chosen sequence of tasks*. Consider the sequence of key tasks presented in this lesson:

1. What is the square of 3?
2. What is the square of -3?
3. What is the positive value of a if the square of a is 9?
4. What is the negative value of a if the square of a is 9?
5. What is the square root of the square of -4?

6. What is the square root of -4?

In each case, procedures were demonstrated by the teacher or suggested by students. But each of these addresses a slightly different issue with respect to squaring and finding square roots. The different issues can be thought of as different conditions under which squaring and finding square roots can operate. The question that is addressed implicitly is how procedures need to be modified to fit the changing conditions.

From a mathematical point of view, the problems selected cover the key categories of conditions that affect the sign-value of squares and square roots. It is as if the territory is being mapped out as each problem explores a slightly different area of the terrain. Each problem adds another section to the map. If students learn procedures to handle each case, they will learn the repertoire of procedures needed to deal with all common cases.

But the procedures for each case or condition are not presented as isolated algorithms. As the lesson proceeds, the procedures accumulate and build. References are made to earlier procedures as later ones are demonstrated. The teacher's aim seems to be to connect procedures to each other as they are developed in turn. Consider the questions asked about how the procedures are similar and different (e.g., at 00:07:03:28 the teacher asks, "What is the difference? What is the difference between this part and the- and the previous question?"). Connections among mathematical entities are, of course, a hallmark of conceptual understanding (Brownell, 1935). So, an up-close view of the activity surrounding each problem might suggest a procedural emphasis (in more traditional terms) whereas a more distant view that encompasses the sequence of problems might see a host of conceptual links.

A third ingredient that enables repetition with variation in a way that might lead to conceptual understanding puts a finer point on the previous two descriptions—*basic definitions presented early in the lesson provide an anchor* which keeps the discussion grounded while allowing exploration of boundary conditions. Throughout the lesson, there is a persistent reference back to the meaning of squaring and taking square roots. These definitions provide the criteria necessary for resolving the new problems and, at the same time, hold the different problems (and lesson segments) together.

In the culminating episode, the teacher returns to the definitions several times to justify the procedures for solving the most challenging problems. When discussing the answer to the problem "What is the square root of the square of -4?" the teacher points to the definition of squaring a number:

```
00:23:17:05 T     Negative- the square of negative four is equal to negative
four times negative four.
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When working through the final problem, "What is the square root of -4?", the teacher returns to the definition of square root:

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00:25:24:03 T     Why?  Because take square root means- what is the number
times itself equal to negative four?
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And when explaining why negative two cannot be the solution, the teacher restates the definition of squaring:

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00:25:49:29 T      Why?  Because negative two times negative two... equals...?
00:25:56:04 SS     Four.
00:25:56:22 T      Four.  So it does not equal negative four.
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In the summary of the lesson, the teacher again returns to the anchor points—the basic definitions that guided the procedures demonstrated during the lesson.

Implications for Reconceptualizing Procedural Knowing

Before reviewing what has been learned from the lesson just presented, it is important to acknowledge that this kind of lesson is not unique to Hong Kong SAR. Similar lessons probably are taught in many countries, including the United States (although no U.S. lessons with this character were found in the TIMSS Video sample). What makes Hong Kong SAR an interesting case is that these kinds of lessons are so common in the TIMSS Video sample of this country, and Hong Kong SAR students perform so well across a range of problem types on international tests.

We propose that the traditional connotations of procedural (and conceptual) knowing, as they have been applied to classroom practice in the United States, do not adequately describe the activity of the lesson presented here. The dichotomy between procedural and conceptual does not fit. One can see elements of the traditional definitions by focusing on particular segments but, in general, the interplay between procedural and conceptual is so complex and intertwined that the traditional definitions lose their usefulness.

Researchers could resolve this dilemma by creating new definitions, or modifying old ones, that capture the range of cognitive and pedagogical activities that fit under "procedural" and "conceptual." The problem then is that the boundaries between these terms would become very fuzzy. The implication derived from reviewing the Hong Kong SAR lesson is that the interplay between procedural and conceptual makes clear distinctions between them problematic.

Alternatively, researchers could resolve the dilemma by creating new terms that are more nuanced and in turn capture the range of activities that occur as procedural and conceptual knowing interact. The problem is that a proliferation of new terms exacerbates, rather than lessens, the difficulty of communicating about cognitive and/or pedagogical activity with more clarity and precision. "Procedural reasoning" is a compromise, of sorts, that describes quite well the nature of the activity in the Hong Kong SAR lesson. But creating phrases that combine old terms does not strike us as a completely satisfactory solution.

Regardless of the eventual solution to the terminology problem, we believe that continuing efforts to define relevant constructs will benefit from taking into account the learning goals when describing the procedural or conceptual character of classroom activity. For example, one can teach or learn procedures for the purpose of correctly and efficiently executing procedures. Or, one can teach or learn procedures in the service of achieving additional goals—inquiring into the nature of particular concepts, mapping out the mathematical terrain of a topic, studying the connections among mathematical procedures or concepts, and so on. Classroom practices in the United States have a long tradition of working toward the goal of efficient execution but have not explored, in general, ways to achieve other goals through teaching and learning procedures. Describing in detail how "procedural reasoning" can be used to accomplish these goals is a first step toward helping classroom teachers understand a richer and more nuanced view of procedural learning.

A final conjecture is that this aspect of instruction—the ways in which procedural and conceptual knowing interact—might be as important for determining students' learning opportunities as many of the pedagogical issues that currently consume much attention, such as student-centered versus teacher-centered or inquiry based versus direct instruction. Perhaps it is more important for students that mathematical ideas with rich interconnected procedural and conceptual features are made public and available than which pedagogical format is used. We expect that re-examining procedural and conceptual knowing will be useful for both theoretical and practical reasons if researchers can develop the definitions and methods to study more carefully how these ways of knowing interact during classroom activity.

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[1] From this point, we will use the term "knowing" rather than "knowledge" to signal the fact that we are as interested in the processes that students use to learn mathematics as the outcomes.
[2] For convenience, in this paper Hong Kong SAR is referred to as a country. Hong Kong SAR is a Special Administrative Region (SAR) of the People's Republic of China.