Basic Probability Theory II

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FREC 408

RECAP

- We said the approach to establishing probabilities for events is to
  - Define the experiment
  - List the sample points
  - Assign probabilities to the sample points
  - Determine the collection of sample points contained in the event of interest
  - Sum the sample point probabilities to get the event probability

Next

- **Complementary** Events
- Compound Events
  - **Union** of Events
  - **Intersection** of Events
- General **Additive Rule**
- Additive Rule for **Mutually Exclusive** Events
- **Multiplicative** Law
- **Conditional** Probability and **Independent** Events
- Equally Likely Mutually Exclusive Events

Probability Rules in Book

- **Rule 1. Additive rule for mutually exclusive events.** If two events are mutually exclusive, then the probability that either Event A or Event B occurs is equal to the sum of their probabilities.
  - **P(A or B) = $P(A \cup B) = P(A) + P(B)$** (p 200)
- **Rule 2. The Probability for an experiment in one of a number of equally likely mutually exclusive events.** If an experiment can result in only one of M equally likely events, and that m of these events result in event A, then the probability of Event A is:
  - **$P(A) = m/M$** (p 201)

Probability Rules in Book

- **Rule 3. Probability Relationship for Complementary Events**
  - **$P(A) = 1- P(A^c)$** (p203)
- **Rule 4. The Probability that Both of Two Independent Events A and B occur.** If two events A and B are independent, then the probability that both A and B occur is equal to the product of their respective unconditional probabilities:
  - **P(A and B) $=P(A \cap B) = P(A)*P(B)$** (p214)

Probability Rules in Book

- **Rule 5. The Multiplicative Law of Probability.** The probability that both of two events A and B occur is
  - **P(A and B) $=P(A \cap B) = P(A)*P(B|A)$**
    $= P(B)*P(A|B)$ (p221)
- **Rule 6. The Additive Law of Probability.** The probability that either event A or event B or both occur is:
  - **P(A or B) $= P(A \cup B)$**
    $= P(A)+P(B)- P(A \cap B)$ (p222)

Complementary Events

- The **complement** of an event A is the event that A does not occur – that is all sample points not in Event A
  - Denoted as $\mathbf{A^C}$ or as **A'** in the book
- $P(A) + P(A^c) = 1.0$
- **Probability of Complementary Events:**
  $\mathbf{P(A^c) = 1- P(A)}$ (RULE #3 p203)

Compound Events

- Events can be comprised of several events joined together, and these are called **COMPOUND EVENTS**
- They can be the **UNION** of several events
- Or the **INTERSECTION** of several events

Union of two Events

- The **union** of two events, A and B is the Event that occurs if either A, B, or both occur on a single performance of the experiment (Def4.11 P221)
- We denote the Union as $\mathbf{A \cup B}$
- $\mathbf{A \cup B}$ consists of all the sample points that belong to A or B **or** both.

Intersection of two Events

- The **intersection** of two events, A and B is the Event that occurs if both A and B occur on a single performance of the experiment (Def4.10 P220)
- We denote the Intersection as $\mathbf{A \cap B}$
- $A \cap B$ consists of all the sample points that belong to both A **and** B.

Example using a die toss

- Event A [Toss an even number]
- Event B [Toss a number <= 3]
- What is $\mathbf{A^c}$?
- What is $\mathbf{A \cup B}$?
- What is $\mathbf{A \cap B}$?
- Can you calculate the probability of the complement, union and the intersection of these events?

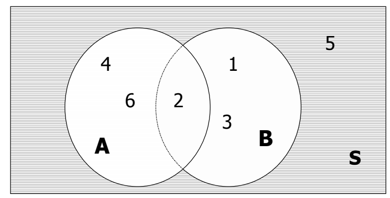
What is A^c for a roll of a die?

- A = [2, 4, 6]
- $\mathbf{A^c}$ = [1, 3, 5]
  - the event A^c is tossing an odd number
  - The probability of this event is 3/6 or ½
  - $P(\mathbf{A^c}) = P(1) + P(3) + P(5) = 3/6 = ½$
- Alternative approach:
  $P(\mathbf{A^c}) = 1- P(A) =1- \{P(2) + P(4) + P(6)\}$
  $= 1 - 3/6 = 1- 1/2 = ½$
  employing Rule#3

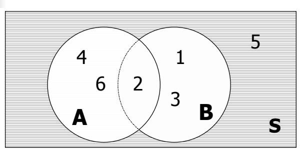
What is $A \cup B$ for a roll of a die?

- $A = [2, 4, 6]$
- $B = [1, 2, 3]$
- $A \cup B =$
  - $[2, 4, 6] + [1, 2, 3] = [1, 2, 3, 4, 6]$
  - **Note that the sample point 2 is common to both events and we don't count it twice**.

What is $A \cup B$ for a roll of a die?



What is $P(A \cup B)$ for a roll of a die?



The probability of this event is 5/6

$P(A \cup B) = P(1)+P(2)+P(3)+P(4)+P(6) = 5/6$

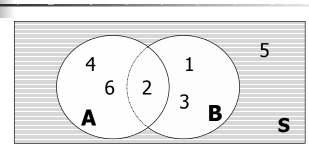
Another way to approach the problem of $A \cup B$

- Find the probability of the **complement**, and subtract from 1
  - $\mathbf{P(A \cup B) = 1 - P(A \cup B)^C}$
- $\mathbf{P(A \cup B)^C}$ would mean everything that wasn't in events A or B
  - In this case it is the value of 5
  - And the probability of rolling a five is 1/6

What is $A \cap B$ for a roll of a die?

- $A = [2, 4, 6]$
- $B = [1, 2, 3]$
- $A \cap B = 2$

What is $P(A \cap B)$ for a roll of a die?



The probability of this event is 1/6

$P(A \cap B) = P(2)$

Additive Rule of Probability (General)

- **P(A ∪ B)**

  = P(A) + P(B) – P(A ∩ B)
- This is called the **Additive Rule of Probability** (RULE#6 P222)

Additive Rule of Probability (in case of mutual exclusive)

- If events A and B are **mutually exclusive**, meaning no intersection, then

  P(A ∪ B)=P(A)+P(B)

  (RULE#1 P200)
- Two events are **mutual exclusive** if when one event occurs in an experiment, the other cannot occur (Def4.4 P198)
  - Example: Events **A** and $\mathbf{A^c}$ are mutual exclusive.

Roulette example

- Roulette is a betting game where a ball spins on a circular wheel that is divided into 38 arcs of equal length
- **Red Numbers**
  - **1 3 5 7 9 12 14 16 18 19 21 23 25 27 30 32 34 36**
- **Black numbers**
  - **2 4 6 8 10 11 13 15 17 20 22 24 26 28 29 31 33 35**
- **Green numbers**
  - **00 0**
- You can bet on odd, even, red, black, high, low

Roulette example

- A: Outcome is an odd number (note 00 and 0 are neither even or odd)
  - A: [1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35]
  - P(A) = 18/38 = .474

Roulette example

- B: Outcome is a black number
  - B: [2,4, 6, 8, 10, 11, 13, 15, 17, 20, 22, 24, 26, 28, 29, 31, 33, 35]
  - P(B) = 18/38 = .474
- C: Outcome is a low number (1-18)
  - C: [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]
  - P(C) = 18/38 = .474

Roulette example

- Find the Intersection of Events A and B
- A ∩ B
  - all numbers that are both odd and black
  - A: [1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35]
  - B: [2,4, 6, 8, 10, 11, 13, 15, 17, 20, 22, 24, 26, 28, 29, 31, 33, 35]

Roulette example

- Find the Intersection of Events A and B
- A ∩ B
  - all numbers that are both odd and black
  - A: [1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35]
  - B: [2,4, 6, 8, 10, 11, 13, 15, 17, 20, 22, 24, 26, 28, 29, 31, 33, 35]

A ∩ B = 11, 13, 15, 17 29, 31, 33, 35

P(A ∩ B) = 8/38 = .211

Roulette example

- Find the Intersection of Events A and C
- A ∩ C
  - all numbers that are both odd and low
  - A: [1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35]
  - C: [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]

Roulette example

- Find the Intersection of Events A and C
- A ∩ C
  - all numbers that are both odd and low
  - A: [**1, 3, 5, 7, 9, 11, 13, 15, 17**, 19, 21, 23, 25, 27, 29, 31, 33, 35]
  - C: [**1**, 2, **3**, 4, **5**, 6, **7**, 8, **9**, 10, **11**, 12, **13**, 14, **15**, 16, **17**, 18]

A ∩ C = **1, 3, 5, 7, 9, 11, 13, 15, 17**

P(A ∩ C) = **9/38 = .237**

Roulette example

- Find the Intersection of Events B and C
- B ∩ C
  - all numbers that are both black and low
  - B: [**2,4, 6, 8, 10**, **11, 13, 15, 17**, 20, 22, 24, 26, 28, 29, 31, 33, 35]
  - C: [1, **2**, 3, **4**, 5, **6**, 7, **8**, 9, **10**, 11, 12, 13, 14, 15, 16, 17, 18]

B ∩ C = 2, 4, 6, 8, 10, 11, 13, 15, 17

P(B ∩ C) = 9/38 = .237

Roulette example

- What about the Intersection between all three Events
- A ∩ B ∩ C
- All odd numbers that are Black and are low
- 11, 13, 15, 17
- P(A ∩ B ∩ C) = 4/38 = .105

Roulette example

- What about the union between two events?
- A ∪ B
- This is all the numbers that are odd or are black
- Note: there is overlap between them, and they are not mutually exclusive

Roulette example

- All odd numbers **and** all that are Black
- 1, 3, 5, 7, 9**, 11, 13, 15, 17**, 19, 21, 23, 25, 27, **29, 31, 33, 35**, **2**, 4, 6, 8 10, 20, 22, 24, 26, 28
- **The green numbers (also in bold) represent the overlap**
- $P(A \cup B) = 28/38 = .737$

Roulette example

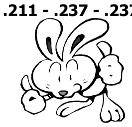
- **We could also use the additive rule**
- **$P(A \cup B) = P(A) + P(B) - P(A \cap B)$**
- $P(A \cup B) = .474 + .474 - .211$
- $P(A \cup B) = .737$
- Note: Because A and B are **not** mutual exclusive, we must use General Additive Rule.

What about $P(A \cup B \cup C)$?

- All points that are either odd, Black, or low
- **1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 2, 4, 6, 8 10, 20, 22, 24, 26, 28, 12, 14, 16, 18**
- **32/38 = .842**

What about $P(A \cup B \cup C)$?

- We can also use the additive rule
- Only this time it's a bit more complicated
- **$P(A \cup B \cup C) =$**
  - **$P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$**
- **= .474 + .474 + .474 - .211 - .237 - .237 + .105**
- **= .842**



So what's the point??

- Not to make you better gamblers
- I do want you to understand
  - that probability is based on events within a sample space
  - that defining the probability of an event(s) can be complicated, particularly for compound events
  - There may be more than one way to solve the problem – counting, using the formulas and rules, using complements

Conditional Probability

- If we have knowledge that affects the outcome of an experiment, the probabilities will be altered
- We call this a **Conditional Probability** (Def4.8 P212)
  - Designated as **$P(A|B)$**
  - The probability of Event A is conditioned on the probability of Event B
- We often use the term **"given"** when talking about conditional probabilities

Conditional Probability

- Suppose we have the roll of a die
  - **P(even number) = P[2, 4, 6] = 3/6 = .5**
- What if we ask the probability of of an even number ***given*** the die is less than or equal to 3?
  - **P(even|≤3) = P[2|≤3] = 1/3**
  - Note: it is a 2 out the new or *given* possible space [1,2,3]

Conditional Probability

- The formula of a conditional probability is:
- Probability of the **intersection of A and B divided by the probability of B**

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

- **It adjusts the probability of the intersection to the reduced sample space of the condition**

Conditional Probability

- Back to the die example
- If A = [even number on a die]
- B = [less than or equal to 3]
- P(A∩B) = P(2) = 1/6 = .1667
- P(B) = P(1) + P(2) + P(3) = 3/6 = .5
- P(A|B) = .1667/.5 = **.333**
  or **1/3**

Next: Multiplicative Rule

- The multiplicative rule shows us the probability of an intersection
- Remember we said a conditional probability is determined by the formula

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Rearrange terms and we can find the formula for the probability of an intersection between A and B

Multiplicative Rule

- This show the probability of an intersection
- It suggests that the probability of an **intersection** between two events depends upon the conditional probability between the two events

$$P(A \cap B) = P(B)P(A|B)$$
$$= P(A)P(B|A)$$

- This is **Multiplicative Rule** (Rule#5 P221).

Probability of an Intersection (Multiplicative Rule)

$$P(A \cap B) = P(B)P(A|B)$$

$$P(B \cap A) = P(A)P(B|A)$$

Multiplicative Rule and Independence

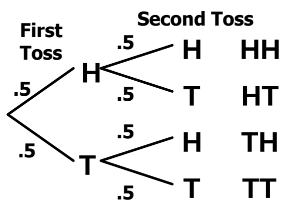
- **If Events A and B are independent of each other**
  - Then, P(A|B) = P(A)
  - Likewise, P(B|A) = P(B) (Def4.9 P214)
- Independence means that the probability of A doesn't change given the event B occurs or not

Multiplicative Rule in case of Independence

- In the case of **independence** between events A and B
- The formula for probability of an interaction reduces to:
  - **$P(A \cap B) = P(A) \cdot P(B)$** (Rule#4 P214)
- **If we can assume independence** between events, figuring the probability of the intersection of events is much easier

Look at the Probability of tossing two coins

What is the probability of getting two heads?



Earlier we looked at this problem differently

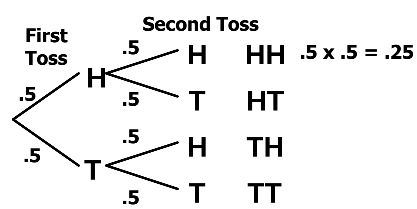
- Sample space for flipping two coins and noting the face:
  - Observe H H ¼ = .25
  - Observe H T ¼ = .25
  - Observe T H ¼ = .25
  - Observe T T ¼ = .25

Use Multiplicative Rule In case of Independence

- We know the first flip is **independent** to the second flip
- The probability of observing a head in a single flip of a coin is ½ or .5
- The probability of observing a tail in a single flip of a coin is ½ or .5
- If I can assume independence
  - P(two Heads) =
  - = P(Head 1st flip) · P(Head on the 2nd flip)
  - **= (.5) ·(.5) = .25**

Multiple through to get the probabilities

What is the probability of getting two heads?



Conditional Probability versus Independence

- Conditional probability and independence are very important concepts in research
  - If we hypothesize salary levels differ between men and women, in essence we are saying, "given you are a female, I expect your salary is different."
  - If we hypothesize that level of response is different between a drug and the treatment group, we are saying, "given you received the drug, your response is higher"
- We often test conditional probability by comparing the data we observe to a hypothetical model of independence

A little circular?

Probability of A Union $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Conditional Probability $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$

Probability of an Intersection $P(A \cap B) = P(B)P(A \mid B)$

Example

- Deck of Cards
- Four suits, Hearts, Diamonds, Clubs, Spades
- 13 in each suit
- Let **Event A =** Face Cards (Jack, Queen, King)
- Let **Event B =** Hearts
- Let **Event C =** Red suit (Hearts, Diamonds)

Example

- Define Experiment: selecting from a single Deck of cards
- List the sample points
  - **Event A** = [Face Cards] = [**J♥ Q♥ K♥;** J♣ Q♣ K♣; **J♦ Q♦ K♦**; J♠ Q♠ K♠]
  - **Event B** = [Hearts] = **[A ♥ 2 ♥ 3 ♥ 4 ♥ 5 ♥ 6 ♥ 7 ♥ 8 ♥ 9 ♥ 10 ♥ J ♥ Q ♥ K ♥]**
  - **Event C =** [Red Suits] = **[A ♥ 2 ♥ 3 ♥ 4 ♥ 5 ♥ 6 ♥ 7 ♥ 8 ♥ 9 ♥ 10 ♥ J ♥ Q ♥ K ♥ ; A ♦ 2 ♦ 3 ♦ 4 ♦ 5 ♦ 6 ♦ 7 ♦ 8 ♦ 9 ♦ 10 ♦ J ♦ Q ♦ K ♦]**

Assign Probabilities

- A random draw of each card is 1/52
- P(Event A) =
  - 12* 1/52 = 12/52 = .2308
- P(Event B) =
  - 13 * 1/52 = 13/52 = .25
- P(Event C) =
  - 26 * 1/52 = 26/52 = .5

What is the Probability of:

- $P(A \cup B)$
- $P(B \cup C)$

What is the Probability of:

- P(A ∩ B)

- P(B ∩ C)

What is the Probability of:

- P(A|B)

Probability Problem

- Enterococci are bacteria that cause blood infections in hospitalized patients. One antibiotic used to battle enterococci is vancomycin. A study by the Federal Centers for Disease Control and Prevention revealed that **8%** of all enterococci isolated in hospitals nationwide were resistant to vancomycin.

Probability Problem

- Consider a random sample of three patients with blood infections caused by the enterococci bacteria. Assume that all three patients are treated with the antibiotic vancomycin.
- What is the probability that all three patients are successfully treated? What assumption did you make concerning the patients?

What is the probability that the bacteria resist the antibiotic for at least one patient?