Basic Probability Theory I

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FREC 408

A Probability puzzler!!

Our Strategy with Probability

- Generally, we want to get to an inference from a sample to a population.
- In this case the population is **unknown**, and we want to infer something from a known sample
- We want the sample to tell us something about the population

In probability

- With probability we will do the **reverse**:
  - We start with the notion that the population is known, i.e., the probabilities
  - And infer something about the chances of obtaining various samples from the population

Some Terms

- If we flip a coin and record the result – a tails
- The result we record is an **observation**
- The process of making an observation is called an **experiment** **(Def4.1 P193)**
  - More specifically, an experiment leads to a **single outcome**
  - Which **can't be predicted with certainty**

Some terms

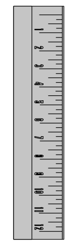
- The basic outcome of an experiment is called a **sample point**
- And the collection of outcomes is called the **sample space** **(Def4.6 P199).**
- Outcomes of experiments are called **events** **(Def4.2 P194).**
- Part of our strategy will be to identify all the sample points, or all the possible outcomes of the experiment
- That is, to identify the sample space

What Does Probability Mean?

- The probability of an event is a proportion between 0 and 1 that **measures the likelihood** that the **outcome will occur** when the experiment is performed.
- If we flip a coin, we expect that the probability of getting a heads is .5
  - *Assuming a fair coin!*

Probabilities

- It is a numerical measure of the likelihood that Outcome A will occur
  - *P*(A)
  - *Prob*(A)



A closer look at probability

- Suppose you flip a coin
  - What's the chance of getting a head?
  - What's the chance of getting a tails?

Flipping a coin

- What does it mean when we state a coin flip as a probability?
  - Does it mean that every other time it will be a heads?
  - If we flip it 10 times, will we always get exactly five heads?
  - If we flip it 1,000 times, will we get exactly 500 heads?

Flipping a coin

- TRY IT IN CLASS
  - Flip a coin 10 times
  - Record the number of HEADs
  - Calculate a probability of a HEAD based on your data
- Calculate the results # Head/10

The story of the Law of Averages

- John Kerrich was a South African mathematician who spent WWII in an internment camp – **very unlucky!**
- He had a lot of time on his hands, so he carried out experiments in probability theory
- One viewpoint about the **Law of Averages** was that if you tossed a coin a lot of times, the number of heads and the number of tails would **eventually equal out**

Law of Averages

- Not so, said Kerrich!
- The probability of every toss is still 50/50. No matter what happened before, each toss has an equal chance
- In 100 tosses of a coin, 1,000 tosses, or 10,000 tosses, the probability of the next toss is still the same.

Law of Averages

- Thus the difference between the number of heads and the number of tails could be quite large in 10,000 tosses
  - We should expect 5,000 heads and 5,000 tails
  - It could easily be 4,900 heads and 5,100 tails

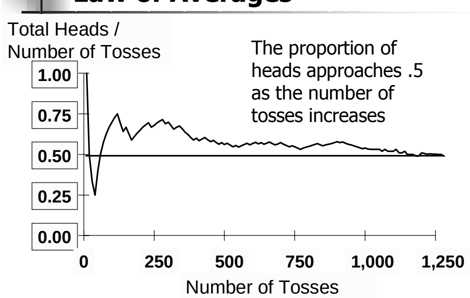
Law of Averages

- But, in relative terms, the difference will get **smaller and smaller** as the number of tosses gets **larger and larger**
- As the number of tosses increases the percentage of heads will approach 50%

Law of Averages

- **Our expectation is 50% heads and tails, with some chance error.**
- As we increase the number of tosses, the observed percentage of heads (or tails) approaches 50%
  - but it may never get there
  - And the ***absolute difference*** between heads and tails may be large

Law of Averages



Probability

- The basic definition of probability is:

$$\frac{\#\ \text{favorable events}}{\text{Total}\ \#\ \text{outcomes}}$$

- It is a **proportion !**

Two Kinds of Probability

- ***A priori*** – given by a definition and before the fact. It is generally mathematically defined.
- **Example:** Rolling a die with equal probabilities for each outcome
  - There are 6 outcomes – the probabilities for rolling a one is **1/6**
  - A priori probabilities are based on the **long run**

Two kinds of probabilities

- ***A posteriori*** – derived empirically through repeated experiments
  - We observe the probabilities after the fact from an experiment or series of experiments
  - A survey approach would be *a posteriori*

An example of an experiment

- Flip two coins in succession, then observe the face of the two coins

Write out the sample space

- Sample space:
  - Observe H H
  - Observe H T
  - Observe T H
  - Observe T T
- S: [HH, HT, TH, TT]

Another example

- We conduct an experiment and roll a single die and record the face

Sample space of rolling a die

- **Sample space:**
  - Observe a 1
  - Observe a 2
  - Observe a 3
  - Observe a 4
  - Observe a 5
  - Observe a 6
- Set Notation S: [1, 2, 3, 4, 5, 6]

The sample space could be defined in several ways

- Experiment: roll a die and observe odd or even number
- Sample Space
  - Observe an even number (2, 4, 6)
  - Observe an odd number (1, 3, 5)
- S: [Even, Odd]

More examples

EXPERIMENT	SAMPLE SPACE
▪Toss a Coin, Note Face	Head, Tail
▪Toss 2 Coins, Note Faces	HH, HT, TH, TT
▪Select 1 Card, Note Kind	2♥, 2♦, ..., A♠ (52)
▪Inspect a Part, Note Quality	Defective, OK
▪Interview and Observe Gender	Male, Female

Outcomes of Experiments

- We want outcomes (i.e., sample points) to be
  - **Mutually Exclusive** – two outcomes can't occur at the same time **(Def4.4 P198)**
  - **Collectively Exhaustive** – all possible outcomes are identified and no more possible outcomes are left out of sample space.

Some more terms

- **Random Trials**
  - The selection of any outcome is not predetermined; thus each outcome has a chance to be selected
  - A fixed uneven dice would not be random, and would be lead to outcomes that are **biased**

More terms

- **Independent Trials**
  - The outcome is not conditioned upon previous events. The outcome of previous event has no compact on the outcome of current event.
  - Flips of a coin are independent – the probability of a head on a second flip is not influenced by the outcome on the first flip

Probability Rules for Sample Points

- All sample point probabilities ***must*** lie between 0 and 1
  - A probability of 1 means **CERTAINTY**
- The probabilities of all sample points within a sample space ***must*** sum to 1

Roll of a single die

Sample Points	Probabilities
1	1/6 = .1667
2	1/6 = .1667
3	1/6 = .1667
4	1/6 = .1667
5	1/6 = .1667
6	1/6 = .1667
TOTAL	6/6 = 1.000

More Terms - Events

- An **event** is a specific collection of sample points
- For example, on a roll of the dice
  - Sample points are [1, 2, 3, 4, 5, 6]
  - The Event A: [Even numbers] contains the sample points 2, 4, 6
  - The Event B: [Odd numbers] contains the sample points 1, 3, 5
- Events are subsets, which can overlap or be mutually exclusive

Probability of an Event

- The **probability of an Event A, denoted as P(A),** is calculated by
- summing the probabilities of the sample points in the sample space for Event A

Steps for Calculating Probabilities of Events

1. **Define** the **experiment** – describe process of making an observation
2. **List** the **Sample Points**
3. **Assign probabilities** to the **Sample Points**
4. Determine the collection of **Sample Points** contained in the **Event** of interest
5. **Sum** the **Sample Points probabilities** to get the **Event** probability

Rolling a single die

- Outcomes U: [1, 2, 3, 4, 5, 6]
- Probabilities for the outcomes
  - Outcome 1 1/6
  - Outcome 2 1/6
  - Outcome 3 1/6
  - Outcome 4 1/6
  - Outcome 5 1/6
  - Outcome 6 1/6

Rolling a single die

- **Event:** rolling a 2
  - Outcome 2
  - P(rolling a 2) = 1/6
- **Event:** rolling 3 or higher
  - Outcomes 3, 4, 5, 6
  - P(rolling 3+) = 1/6+ 1/6 + 1/6 +1/6 = 2/3
  - P(rolling 3+) = .667

Rolling a single die

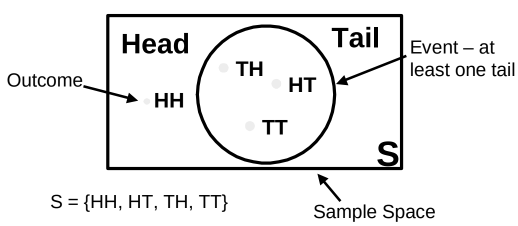
- **Event:** rolling an even number
  - Outcomes 2, 4, 6
  - P (rolling even) = 1/6 +1/6 +1/6 = .5

Representing the Sample Space or Events

- We can represent the Sample Space and Events by
  - Listing of the Set
  - Venn Diagram
  - Contingency Table
  - Decision Tree Diagram

Venn Diagram

Experiment: Toss 2 Coins. Note Faces.



S = {HH, HT, TH, TT}

Contingency Table

Experiment: Toss 2 Coins. Note Faces.

	2nd Coin		
1st Coin	**Head**	**Tail**	**Total**
Head	**HH**	**HT**	**HH, HT**
Tail	**TH**	**TT**	**TH, TT**
Total	**HH, TH**	**HT, TT**	**S**

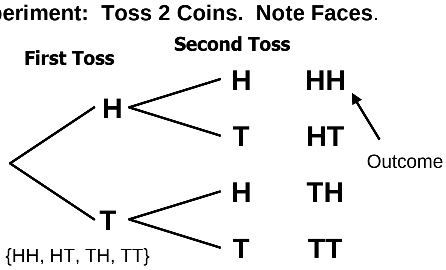
Simple Event (Head on 1st Coin) → Head

Sample Space

S = {HH, HT, TH, TT}

Tree Diagram

Experiment: Toss 2 Coins. Note Faces.



S = {HH, HT, TH, TT}

Problem example

- I have a jar containing five marbles, 2 of which are blue and 3 of which are red.
- I randomly draw two marbles
- What is the probability of drawing **2 blue marbles**?

Marble problem solution

- Define the experiment
- List or draw out the sample points
- Assign probabilities to the sample points
- Determine the collection of sample points contained in an event of interest
- Sum the sample points probabilities to get the event probability

Marble problem

- Sample Points **B for blue R for Red**
  - B_1 B_2
  - B_1 R_1
  - B_1 R_2
  - B_1 R_3
  - B_2 R_1
  - B_2 R_2
  - B_2 R_3
  - R_1 R_2
  - R_1 R_3
  - R_2 R_3

Marble Problem

- In this problem the order of the marbles is not important to us
- So there is only one combination of drawing two blue marbles
- Unless otherwise known, each sample point has an **equal probability**
- Thus each has a 1/10 chance of being drawn

Marble problem

Sample Points	**Probability**
B_1 B_2	1/10
B_1 R_1	1/10
B_1 R_2	1/10
B_1 R_3	1/10
B_2 R_1	1/10
B_2 R_2	1/10
B_2 R_3	1/10
R_1 R_2	1/10
R_1 R_3	1/10
R_2 R_3	1/10

Marble problem

- What is the probability that two blue marbles are drawn?
  - 1/10
- What is the probability that a blue and a red marble are drawn?
  - 6/10 OR 3/5
- What is the probability that two red marbles are drawn
  - 3/10

Express sample points as combinations of blue and red marbles

- Sample Points Probability
  - Two Blue
  - Blue & Red
  - Two Red

Express sample points as combinations of blue and red marbles

- Sample Points — Probability
  - Two Blue — 1/10
  - Blue & Red — 6/10
  - Two Red — 3/10

Combinatorial formula

- To find the number of samples of N things taken n at a time

$$\binom{N}{n} = \frac{N!}{n!(N-n)!}$$

- Where n! is

$$n! = n(n-1)(n-2)...(1)$$

$$5! = 5(5-1)(5-2)(5-3)(5-4) = 120$$

- Note: 0! = 1 and 1! = 1

Combinatorial formula

- For our problem we have 5 marbles and two outcomes
- N=5 and n=2

$$\binom{5}{2} = \frac{5!}{2!(5-2)!}$$

$$\frac{120}{2(6)} = \frac{120}{12} = 10$$

Survey Problem

- A researcher wanted to find the primary reason for a company to engage in diversity training
- She surveyed businesses to determine the primary reason for diversity training, offering five ***mutually exclusive***, and ***exhaustive*** options
- Listed the percentages for each

Diversity survey problem

Reason	Percentage
Comply with personnel policies (CPP)	7%
Increase productivity (IP)	47%
Stay competitive (SC)	38%
Social responsibility (SR)	4%
Other (O)	4%

The percentages can be read as probabilities

Diversity problem

- What is the probability that the primary reason for diversity training is business related; i.e., related to competition or productivity?
- What is probability that social responsibility is not a primary reason for diversity training?

Diversity problem

- What is the probability that the primary reason for diversity training is business related; i.e., related to competition or productivity?
- P(B) = P(IP+SC) = .47 + .38 = .85

- What is probability that social responsibility is not a primary reason for diversity training?
- P(Not SR) = P(CPP+IP+SC+O) =
  - .07 + 47 + .38 + .04 = .96

Diversity Problem

- For the last answer
  - We could have thought of this as the compliment of SR
  - Denoted as SR^C
  - SR^C = 1 – P(SR)
  - SR^C = 1 - .04 = .96